On the flow of an elastico-viscous liquid in a curved pipe under a pressure gradient

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Consideration is given to the flow of an idealized elastico-viscous liquid in a curved pipe under a pressure gradient. By using the series expansion method of Dean (1927, 1928) in powers of a/R where a is the radius of the pipe and R the radius of curvature of its 'central line', it is shown that the general nature of the motion is similar to that of the motion of a Newtonian viscous liquid, the liquid elements moving along the pipe in two sets of spirals separated by the central plane. However elasticity of the type considered could strongly affect the pitch of these spirals. To the approximation considered, the flow pattern of the elastico-viscous liquid depends only on the limiting (zero-shear-rate) viscosity η_0 and the first moment, K_0, of the distribution of relaxation times. The corresponding stress components involve also the second moment of this distribution.

It is also shown that the presence of elasticity in the liquid increases the rate of discharge of the liquid.

1. Introduction

During the last fifteen years there has been an increasing interest in the flow behaviour of non-Newtonian fluids, especially fluids that exhibit elasticity in shear, these being known as elastico-viscous liquids. Efforts have been made to characterize these materials by general rheological equations of state, and it is becoming apparent that to do this any complete way requires high-powered mathematics, and the resulting equations are often too complicated to be handled in flow problems other than those involving simple shearing (see, for example, Coleman & Noll 1959, 1961).

Since many of these materials are of industrial importance, it is clearly desirable that the theoretician should be able to characterize them by *simple* equations of state that give an approximate description of observed behaviour and yet are simple enough to be useful in flow problems other than those involving simple shearing (Oldroyd 1962).

When attention is confined to small rates of shear it is well known that many elastico-viscous liquids can be characterized by a spectrum of relaxation times (Walters 1960, 1961). We shall confine ourselves to a consideration of such materials in the present paper. The equations of state for these materials (at small rates of shear) can be written in the form (Walters 1960)†

$$p_{ik} = -pg_{ik} + p'_{ik}, \tag{1}$$

$$p'_{ik} = 2\int_{-\infty}^{t} \Psi(t-t')\, e^{(1)}_{ik}(t')\, dt', \tag{2}$$

where $$\Psi(t-t') = \int_0^\infty \frac{N(\tau)}{\tau} e^{-(t-t')/\tau}\, d\tau,$$

$N(\tau)$ being the distribution function of relaxation times τ. In these equations, p_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} the metric tensor of a fixed co-ordinate system x^i, and $e^{(1)}_{ik}$ is the rate-of-strain tensor.

There are an infinite number of possible sets of equations of state which are valid for all conditions of motion and stress and which reduce to (2) when the rate of strain is restricted to be small (cf. Oldroyd 1950; Walters 1962*a*). Many of these will be too complicated to be useful in general flow problems. A detailed theoretical investigation has recently begun for one of the simplest possible sets of generally valid equations of state. An investigation of this sort is a necessary first step in the study of materials with complicated memory-type equations of state. The equations of state in question, corresponding to the liquid designated B' by Walters (1962*b*), are of the form (1) and

$$p'^{ik}(x,t) = 2\int_{-\infty}^{t} \Psi(t-t') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^r} e^{(1)mr}(x',t')\, dt', \tag{3}$$

where x'^i $(= x'^i(x,t,t'))$ is the position at time t' of the element that is instantaneously at the point x^i at time t.

The liquid designated liquid B by Oldroyd (1950) with equations of state

$$p'^{ik} + \lambda_1 \frac{\mathfrak{d}}{\mathfrak{d}t} p'^{ik} = 2\eta_0\left(1 + \lambda_2 \frac{\mathfrak{d}}{\mathfrak{d}t}\right) e^{(1)ik},\ddagger \tag{4}$$

is a special case of liquid B', obtained by writing

$$N(\tau) = \eta_0(\lambda_2/\lambda_1)\,\delta(\tau) + \eta_0\{(\lambda_1-\lambda_2)/\lambda_1\}\,\delta(\tau-\lambda_1),\S \tag{5}$$

in equations (1) and (3). The Newtonian liquid of constant viscosity η_0 is also a special case, given by

$$N(\tau) = \eta_0\delta(\tau). \tag{6}$$

In the present paper, we shall be concerned with an investigation of the flow of liquid B' through a curved pipe under a pressure gradient. The work was suggested by Dean's treatment of the associated viscous flow problem (Dean 1927, 1928). To the authors' knowledge, no theoretical work has been done on the flow of elastico-viscous liquids through curved pipes, although Jones (1960) has considered the problem for a non-Newtonian visco-*inelastic* liquid.

† Covariant suffixes are written below, contravariant suffixes above, and the usual summation convention for repeated suffices is assumed.

‡ $\mathfrak{d}/\mathfrak{d}t$ is the convected time derivative introduced by Oldroyd (1950).

§ δ denotes a Dirac delta function defined in such a way that

$$\delta(x) = 0,\ (x \neq 0), \quad \int_{-\infty}^{\infty} \delta(x)\, dx = \int_0^\infty \delta(x)\, dx = 1.$$

2. Flow through a curved pipe

It is convenient to consider the motion referred to the co-ordinate system introduced by Dean (1927, 1928); this co-ordinate system is shown in figure 1. OS is the axis of the anchor ring formed by the pipe wall. C is the centre of the section of the pipe by a plane through OS making an angle θ with a fixed axial plane. CO is the perpendicular drawn from C onto OS, and is of length R; R is, therefore, the radius of the circle in which the lines of centres of sections is coiled. The plane through O perpendicular to OS and the line traced out by C will be called the central plane and the central line, respectively, of the pipe. The position of any point P in the section can be specified by the orthogonal co-ordinates, r, ψ, θ; r is its distance from C, and ψ is the angle between CP and a line through C parallel to OS. The surface of the pipe is then given by $r = a$, where a is the radius of any section. The line element dS is given by

$$ds^2 = (dr)^2 + (r\,d\psi)^2 + ([R + r\sin\psi]\,d\theta)^2. \tag{7}$$

We shall suppose that the physical components of the velocity vector corresponding to these co-ordinates are U, V, W, and that the general direction of flow is the direction in which θ increases.

As in the case considered by Dean, we shall suppose that the motion of the liquid is due to a fall of pressure along the pipe. The differential equations of motion relating the physical components of the partial stress tensor $p'_{(ik)}$, the pressure p and the acceleration, for a steady motion in which U, V, W (but not p) are independent of θ, are

$$\left.\begin{aligned}
&\rho\left[\frac{U\,\partial U}{\partial r} + \frac{V\,\partial U}{r\,\partial\psi} - \frac{V^2}{r} - \frac{W^2\sin\psi}{(R + r\sin\psi)}\right] = -\frac{\partial p}{\partial r} + \frac{\partial\{r(R + r\sin\psi)\,p'_{(rr)}\}}{r(R + r\sin\psi)\,\partial r}\\
&\qquad\qquad + \frac{\partial\{(R + r\sin\psi)\,p'_{(r\psi)}\}}{r(R + r\sin\psi)\,\partial\psi} - \frac{p'_{(\psi\psi)}}{r} - \frac{p'_{(\theta\theta)}\sin\psi}{(R + r\sin\psi)},\\
&\rho\left[\frac{U\,\partial(rV)}{r\,\partial r} + \frac{V\,\partial V}{r\,\partial\psi} - \frac{W^2\cos\psi}{(R + r\sin\psi)}\right] = -\frac{\partial p}{r\,\partial\psi} + \frac{\partial\{(R + r\sin\psi)\,p'_{(\psi\psi)}\}}{r(R + r\sin\psi)\,\partial\psi}\\
&\qquad\qquad + \frac{\partial\{r^2(R + r\sin\psi)\,p'_{(r\psi)}\}}{r^2(R + r\sin\psi)\,\partial r} - \frac{p'_{(\theta\theta)}\cos\psi}{(R + r\sin\psi)},\\
&\rho\left[\frac{U\,\partial\{(R + r\sin\psi)\,W\}}{(R + r\sin\psi)\,\partial r} + \frac{V\,\partial\{(R + r\sin\psi)\,W\}}{r(R + r\sin\psi)\,\partial\psi}\right]\\
&\qquad = -\frac{\partial p}{(R + r\sin\psi)\,\partial\theta} + \frac{\partial\{r(R + r\sin\psi)^2\,p'_{(r\theta)}\}}{r(R + r\sin\psi)^2\,\partial r} + \frac{\partial\{(R + r\sin\psi)^2\,p'_{(\psi\theta)}\}}{r(R + r\sin\psi)^2\,\partial\psi};
\end{aligned}\right\} \tag{8}$$

and the equation of continuity is

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{U\sin\psi}{(R + r\sin\psi)} + \frac{V\cos\psi}{(R + r\sin\psi)} + \frac{\partial V}{r\,\partial\psi} = 0. \tag{9}$$

The full equations of state relating the physical components of partial stress and velocity are somewhat complicated; these equations must be solved in conjunction with (8) and (9) and the boundary conditions on U, V, W. As it is difficult to obtain a general solution of all the equations, we shall make an assumption regarding the curvature of the pipe, namely that the quantity a/R is small. Then it is possible to replace

$$1/(R+r\sin\psi) \quad \text{by} \quad 1/R$$

and $\partial/\partial r \pm \sin\psi/(R+r\sin\psi)$, $\partial/r\partial\psi \pm \cos\psi/(R+r\sin\psi)$ by $\partial/\partial r$, $\partial/r\partial\psi$,

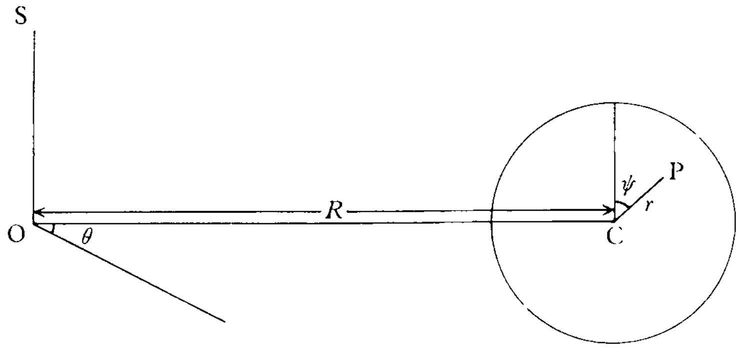

FIGURE 1. The co-ordinate system.

respectively. Also, we shall neglect $e^{(1)\theta\theta}$; because this is at most of order a/R† and is seen to occur only in the expression for $p'_{(\theta\theta)}$ which is itself divided by R in the stress equations of motion. Essentially, these are the approximations introduced by Dean (1927). The equations of motion and continuity under these conditions reduce to

$$\rho\left[\frac{U\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial\psi}-\frac{V^2}{r}-\frac{W^2\sin\psi}{R}\right] = -\frac{\partial p}{\partial r}+\frac{\partial(rp'_{(rr)})}{r\partial r}+\frac{\partial p'_{(r\psi)}}{r\partial\psi}-\frac{p'_{(\psi\psi)}}{r}-\frac{p'_{(\theta\theta)}\sin\psi}{R}, \tag{10}$$

$$\rho\left[\frac{U\partial(rV)}{r\partial r}+\frac{V\partial V}{r\partial\psi}-\frac{W^2\cos\psi}{R}\right] = -\frac{\partial p}{r\partial\psi}+\frac{\partial p'_{(\psi\psi)}}{r\partial\psi}+\frac{\partial(r^2p'_{(r\psi)})}{r^2\partial r}-\frac{p'_{(\theta\theta)}\cos\psi}{R}, \tag{11}$$

$$\rho\left[\frac{U\partial W}{\partial r}+\frac{V}{r}\frac{\partial W}{\partial\psi}\right] = -\frac{\partial p}{R\partial\theta}+\frac{\partial(rp'_{(r\theta)})}{r\partial r}+\frac{\partial p'_{(\psi\theta)}}{r\partial\psi}, \tag{12}$$

$$\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{\partial V}{r\partial\psi} = 0. \tag{13}$$

As U, V, W are assumed independent of θ it follows that the partial stresses $p'_{(ik)}$ are independent of θ; then, from equation (12), we have that p is of the form $\theta f_1(r,\psi)+f_2(r,\psi)$; finally, from (10) and (11), it follows that f_1 must be a constant. Following Dean (1927), we write

$$-R^{-1}\partial p/\partial\theta = G, \tag{14}$$

where G is a constant mean pressure gradient—the space-rate of decrease of p along the central line.

† $e^{(1)\theta\theta}$ is zero for flow along a straight pipe, and so is at most of order a/R during flow through a curved pipe.

It is possible to write equations (10) to (13) in non-dimensional form by using the following substitutions:

$$\left.\begin{aligned} &U = \nu u/a, \quad V = \nu v/a, \quad W = W_0 w, \quad r = ar_1, \quad p = (\eta_0 \nu/a^2)p^*, \\ &p'_{(ik)} = \frac{\eta_0 \nu}{a^2}\begin{bmatrix} p''_{(r_1r_1)} & p''_{(r_1\psi)} & (W_0 a/\nu)\,p''_{(r_1\theta)} \\ p''_{(r_1\psi)} & p''_{(\psi\psi)} & (W_0 a/\nu)\,p''_{(\psi\theta)} \\ (W_0 a/\nu)\,p''_{(r_1\theta)} & (W_0 a/\nu)\,p''_{(\psi\theta)} & (W_0 a/\nu)^2 p''_{(\theta\theta)} \end{bmatrix}, \end{aligned}\right\} \tag{15}$$

where W_0 has the dimensions of a velocity and

$$\nu = \eta_0/\rho, \quad \eta_0 \left(= \int_0^\infty N(\tau)\,d\tau\right)$$

being the limiting viscosity at small rates of shear (Walters 1960). Equations (10) to (13) then become

$$\begin{aligned} \frac{u\,\partial u}{\partial r_1} + \frac{v\,\partial u}{r_1\,\partial\psi} - \frac{v^2}{r_1} - \tfrac{1}{2}Lw^2\sin\psi \\ = -\frac{\partial p^*}{\partial r_1} + \frac{1}{r_1}\frac{\partial}{\partial r_1}(r_1 p''_{(r_1r_1)}) + \frac{\partial p''_{(r_1\psi)}}{r_1\,\partial\psi} - \frac{p''_{(\psi\psi)}}{r_1} - \tfrac{1}{2}Lp''_{(\theta\theta)}\sin\psi, \end{aligned} \tag{16}$$

$$\frac{u\,\partial(r_1 v)}{r_1\,\partial r_1} + \frac{v\,\partial v}{r_1\,\partial\psi} - \tfrac{1}{2}Lw^2\cos\psi = -\frac{\partial p^*}{r_1\,\partial\psi} + \frac{\partial p''_{(\psi\psi)}}{r_1\,\partial\psi} + \frac{\partial(r_1^2 p''_{(r_1\psi)})}{r_1^2\,\partial r_1} - \tfrac{1}{2}Lp''_{(\theta\theta)}\cos\psi, \tag{17}$$

$$\frac{u\,\partial w}{\partial r_1} + \frac{v\,\partial w}{r_1\,\partial\psi} = C + \frac{\partial(r_1 p''_{(r_1\theta)})}{r_1\,\partial r_1} + \frac{\partial p''_{(\psi\theta)}}{r_1\,\partial\psi}, \tag{18}$$

$$\frac{\partial u}{\partial r_1} + \frac{u}{r_1} + \frac{\partial v}{r_1\,\partial\psi} = 0, \tag{19}$$

where it has been convenient to define (cf. Dean 1927, 1928)

$$L = 2W_0^2 a^3/\nu^2 R, \quad C = Ga^2/\eta_0 W_0. \tag{20}$$

Following Dean, we suppose that W_0 is the θ-component of the velocity at any point of the central line, in the case of slow motion; and in this case, the distribution of the θ-component of velocity approximates to that occurring in the straight-pipe problem, where it is seen to be parabolic (Walters 1962*c*). $W_0 a/\nu$ is then approximately equal to the Reynolds number n—defined as $2\bar{v}a/\nu$, where $\bar{v}$ is the mean velocity over the section. For slow motion it follows that $L = 2n^2a/R$. To find the solution to the problem of flow through a curved pipe, we must solve equations (3), (16) to (19) with the boundary conditions

$$u = v = w = 0 \quad \text{on} \quad r_1 = 1. \tag{21}$$

The method of solution given below is one of successive approximation in which it is assumed that u, v and w can be expanded in ascending powers of L.

When the pipe is straight, a/R and L are zero, and it can be easily shown that equations (3), (16) to (19) have a simple solution. In this case $u = v = 0$ and (18) reduces to

$$0 = C + \nabla_1^2 w. \tag{22}$$

Equation (22) and the boundary condition on w are satisfied by

$$w = 1 - r_1^2, \tag{23}$$

provided $C = 4$; the associated non-dimensional stress components are these:

$$\left.\begin{aligned} p''_{(r_1r_1)} &= 0; \quad p''_{(\psi\psi)} = 0; \quad & p''_{(r_1\psi)} &= 0; \\ p''_{(\psi\theta)} &= 0; \quad p''_{(\theta\theta)} = 8mr_1^2; \quad & p''_{(r_1\theta)} &= -2r_1; \end{aligned}\right\} \tag{24}$$

where $$m = K_0/\rho a^2 \quad \text{and} \quad K_0 = \int_0^\infty \tau N(\tau)\,d\tau.\dagger$$

When the pipe is curved, and a/R and L are sufficiently small, we assume that

$$\left.\begin{aligned} u &= Lu_1 + L^2u_2 + \dots, \\ v &= Lv_1 + L^2v_2 + \dots, \\ w &= (1-r_1^2) + Lw_1 + L^2w_2 + \dots, \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} p''_{(r_1r_1)} &= Lp''_{(r_1r_1)1} + L^2p''_{(r_1r_1)2} + \dots, \\ p''_{(\psi\psi)} &= Lp''_{(\psi\psi)1} + L^2p''_{(\psi\psi)2} + \dots, \\ p''_{(\theta\theta)} &= 8mr_1^2 + Lp''_{(\theta\theta)1} + L^2p''_{(\theta\theta)2} + \dots, \\ p''_{(\psi\theta)} &= Lp''_{(\psi\theta)1} + L^2p''_{(\psi\theta)2} + \dots, \\ p''_{(r_1\theta)} &= -2r_1 + Lp''_{(r_1\theta)1} + L^2p''_{(r_1\theta)2} + \dots, \\ p''_{(r_1\psi)} &= Lp''_{(r_1\psi)1} + L^2p''_{(r_1\psi)2} + \dots. \end{aligned}\right\} \tag{26}$$

In determining the relation between the velocity distribution (25) and the stress distribution (26) from the equations of state (3), it is convenient to work in terms of the original variables in the first instance, using the substitutions (15) later in the analysis.

We write the displacement functions x'^i corresponding to the velocity distribution (25) as

$$\left.\begin{aligned} r' &= r + L\alpha_1(r,\psi,t,t') + L^2\alpha_2(r,\psi,t,t') + \dots, \\ \psi' &= \psi + L\beta_1(r,\psi,t,t') + L^2\beta_2(r,\psi,t,t') + \dots, \\ \theta' &= \theta + \gamma_0(r,\psi,t,t') + L\gamma_1(r,\psi,t,t') + L^2\gamma_2(r,\psi,t,t') + \dots, \end{aligned}\right\} \tag{27}$$

where α_1, β_1, γ_1, etc., are restricted by the conditions that

$$[\alpha_1]_{t'=t} = [\beta_1]_{t'=t} = [\gamma_1]_{t'=t} = 0, \tag{28}$$

etc. The velocity distribution and the displacement functions are related by the equations (cf. Oldroyd 1950, equation (21))

$$\left.\begin{aligned} \frac{\partial r'}{\partial t} + \frac{U\partial r'}{\partial r} + \frac{V}{r}\frac{\partial r'}{\partial \psi} + \frac{W}{R}\frac{\partial r'}{\partial \theta} &= 0, \\ \frac{\partial \psi'}{\partial t} + \frac{U\partial \psi'}{\partial r} + \frac{V}{r}\frac{\partial \psi'}{\partial \psi} + \frac{W}{R}\frac{\partial \psi'}{\partial \theta} &= 0, \\ \frac{\partial \theta'}{\partial t} + \frac{U\partial \theta'}{\partial r} + \frac{V}{r}\frac{\partial \theta'}{\partial \psi} + \frac{W}{R}\frac{\partial \theta'}{\partial \theta} &= 0. \end{aligned}\right\} \tag{29}$$

† For Oldroyd's liquid B (equation (4)), m is given by $m = \eta_0(\lambda_1-\lambda_2)/(\rho a^2)$, and $m = 0$ for the Newtonian liquid.

Substituting (25) and (27) into equations (29) and equating coefficients of L, we obtain the required displacement functions

$$r' = r - \frac{Lvu_1}{a}(t-t') + L^2\left[-\frac{vu_2}{a}(t-t') + \frac{v^2u_1}{a^2}\frac{\partial u_1}{\partial r}\frac{(t-t')^2}{2} + \frac{v^2v_1}{a^2r}\frac{\partial u_1}{\partial \psi}\frac{(t-t')^2}{2}\right], \quad (30)$$

$$\psi' = \psi - \frac{Lvv_1}{ar}(t-t') + L^2\left[-\frac{vv_2}{ar}(t-t') + \frac{v^2u_1}{a^2}\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right)\frac{(t-t')^2}{2} + \frac{v^2v_1}{a^2r^2}\frac{\partial v_1}{\partial \psi}\frac{(t-t')^2}{2}\right], \quad (31)$$

$$\begin{aligned}\theta' = \theta - \frac{W_0}{R}\left(\frac{a^2-r^2}{a^2}\right)(t-t') + L\left[-\frac{vW_0}{Ra^3}u_1 r(t-t')^2 - \frac{W_0 w_1}{R}(t-t')\right] \\ + L^2\left[-\frac{vu_2 rW_0}{a^3R}(t-t')^2 + \frac{v^2u_1^2 W_0(t-t')^3}{Ra^4 3} + \frac{v^2W_0u_1}{Ra^4}\frac{\partial u_1}{\partial r}\frac{r(t-t')^3}{3}\right. \\ \left. + \frac{W_0vu_1}{Ra}\frac{\partial w_1}{\partial r}\frac{(t-t')^2}{2} + \frac{v^2W_0v_1}{Ra^4}\frac{\partial u_1}{\partial \psi}\frac{(t-t')^3}{3} + \frac{vv_1}{arR}\frac{\partial w_1}{\partial \psi}W_0\frac{(t-t')^2}{2} - \frac{W_0w_2}{R}(t-t')\right].\end{aligned} \quad (32)$$

We shall in the first instance work to first order in L.

From (30) to (32), we obtain the following results which are needed in the determination of the stress components:

$$\left.\begin{aligned}&\frac{\partial r}{\partial r'} = 1 + \frac{Lv}{a}\frac{\partial u_1}{\partial r}(t-t'); \quad \frac{\partial \psi}{\partial r'} = \frac{Lv}{a}\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right)(t-t'); \quad \frac{\partial \psi}{\partial \psi'} = 1 + \frac{Lv}{ar}\frac{\partial v_1}{\partial \psi}(t-t');\\ &\frac{\partial r}{\partial \psi'} = \frac{Lv}{a}\frac{\partial u_1}{\partial \psi}(t-t'); \quad \frac{\partial \psi}{\partial \theta'} = 0; \quad \frac{\partial r}{\partial \theta'} = 0; \quad \frac{\partial \theta}{\partial \theta'} = 1;\\ &\frac{\partial \theta}{\partial \psi'} = -\frac{LrvW_0}{Ra^3}\frac{\partial u_1}{\partial \psi}(t-t')^2 + \frac{LW_0}{R}\frac{\partial w_1}{\partial \psi}(t-t');\\ &\frac{\partial \theta}{\partial r'} = -\frac{2rW_0}{Ra^2}(t-t') + \frac{LW_0}{R}\frac{\partial w_1}{\partial r}(t-t') - \frac{LvW_0 r}{Ra^3}\frac{\partial u_1}{\partial r}(t-t')^2 + \frac{LvW_0}{Ra^3}u_1(t-t')^2.\end{aligned}\right\} \quad (33)$$

We now determine the contravariant rate-of-strain components $e^{(1)mr}(r', \psi', t')$ that appear in the equations of state (3), i.e. the rate-of-strain components at time t' in the element that is instantaneously at the point (r, ψ, θ) at time t. These can be obtained by writing down the rate-of-strain components for the element at (r, ψ, θ) at time t, replacing r, ψ, θ, t in these components by r', ψ', θ', t', respectively, and using (30) to (32). In this way, we obtain

$$\left.\begin{aligned}e^{(1)rr}(r', \psi', t') &= e^{(1)rr}(r, \psi, t, t') = \frac{Lv}{a}\frac{\partial u_1}{\partial r},\\ e^{(1)\psi\psi}(r', \psi', t') &= e^{(1)\psi\psi}(r, \psi, t, t') = \frac{Lv}{a}\left[\frac{1}{r^3}\frac{\partial v_1}{\partial \psi} + \frac{u_1}{r^3}\right],\\ e^{(1)\psi\theta}(r', \psi', t') &= e^{(1)\psi\theta}(r, \psi, t, t') = \frac{LW_0}{2Rr^2}\frac{\partial w_1}{\partial \psi},\\ e^{(1)r\theta}(r', \psi', t') &= e^{(1)r\theta}(r, \psi, t, t') = -\frac{W_0 r}{Ra^2} + \frac{LW_0}{R}\left[\frac{1}{2}\frac{\partial w_1}{\partial r} + \frac{vu_1}{a^3}(t-t')\right],\\ e^{(1)r\psi}(r', \psi', t') &= e^{(1)r\psi}(r, \psi, t, t') = \frac{Lv}{a}\left[\frac{1}{2r}\frac{\partial v_1}{\partial r} - \frac{v_1}{2r^2} + \frac{1}{2r^2}\frac{\partial u_1}{\partial \psi}\right].\end{aligned}\right\} \quad (34)$$

Equations (3), (33) and (34) can now be used to determine the physical components of the partial stress tensor. After some reduction, we obtain†

$$\left.\begin{aligned}
&p'_{(rr)} = 2\eta_0\frac{L\nu}{a}\frac{\partial u_1}{\partial r}; \quad p'_{(\psi\psi)} = 2\eta_0\frac{L\nu}{a}\left[\frac{1}{r}\frac{\partial v_1}{\partial \psi}+\frac{u_1}{r}\right]; \quad p'_{(r\psi)} = \frac{\eta_0 L\nu}{a}\left[\frac{\partial v_1}{\partial r}-\frac{v_1}{r}+\frac{1}{r}\frac{\partial u_1}{\partial \psi}\right];\\
&p'_{(r\theta)} = -\frac{2\eta_0 W_0 r}{a^2}+L\left[\eta_0 W_0\frac{\partial w_1}{\partial r}+\frac{K_0 W_0 \nu}{a^3}\left\{2u_1-6r\frac{\partial u_1}{\partial r}\right\}\right];\\
&\qquad\qquad p'_{(\theta\psi)} = L\left[\frac{W_0\eta_0}{r}\frac{\partial w_1}{\partial \psi}+\frac{2K_0 W_0\nu}{a^3}\left\{-2r^2\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right)-\frac{\partial u_1}{\partial \psi}\right\}\right];\\
&p'_{(\theta\theta)} = \frac{8K_0W_0^2r^2}{a^4}+L\left[-\frac{8K_0W_0^2 r}{a^2}\frac{\partial w_1}{\partial r}+\frac{24S_0W_0^2\nu r^2}{a^5}\frac{\partial u_1}{\partial r}-\frac{24S_0W_0^2\nu r u_1}{a^5}\right],
\end{aligned}\right\} \tag{35}$$

where $$S_0 = \int_0^\infty \tau^2 N(\tau)\,d\tau.$$

It is convenient at this stage, after inspection of (13), to introduce a stream function $f(r,\psi)$ defined by

$$rU = -\partial f/\partial\psi, \quad V = \partial f/\partial r. \tag{36}$$

Writing $f = \nu[L\phi_1+L^2\phi_2+\dots]$, we have

$$\left.\begin{aligned}
u_1 &= -\frac{1}{r_1}\frac{\partial\phi_1}{\partial\psi}, \quad v_1 = \frac{\partial\phi_1}{\partial r_1},\\
u_2 &= -\frac{1}{r_1}\frac{\partial\phi_2}{\partial\psi}, \quad v_2 = \frac{\partial\phi_2}{\partial r_1}, \quad \text{etc.}
\end{aligned}\right\} \tag{37}$$

Substituting (37) into (35), and using (15) and (26), we obtain

$$\left.\begin{aligned}
p''_{(r_1r_1)1} &= 2\left(\frac{\partial\phi_1}{r_1^2\partial\psi}-\frac{\partial^2\phi_1}{r_1\partial r_1\partial\psi}\right),\\
p''_{(\psi\psi)1} &= 2\left(\frac{\partial^2\phi_1}{r_1\partial r_1\partial\psi}-\frac{\partial\phi_1}{r_1^2\partial\psi}\right),\\
p''_{(\theta\theta)1} &= 48s\frac{\partial\phi_1}{\partial\psi}-24sr_1\frac{\partial^2\phi_1}{\partial r_1\partial\psi}-8mr_1\frac{\partial w_1}{\partial r_1},\\
p''_{(\psi\theta)1} &= \frac{\partial w_1}{r_1\partial\psi}+2m\left[\frac{\partial^2\phi_1}{r_1\partial\psi^2}-2r_1^2\frac{\partial}{\partial r_1}\left(\frac{\partial\phi_1}{r_1\partial r_1}\right)\right],\\
p''_{(r_1\theta)1} &= \frac{\partial w_1}{\partial r_1}-m\left[8\frac{\partial\phi_1}{r_1\partial\psi}-6\frac{\partial^2\phi_1}{\partial r_1\partial\psi}\right],\\
p''_{(r_1\psi)1} &= -\frac{\partial^2\phi_1}{r_1^2\partial\psi^2}+r_1\frac{\partial}{\partial r_1}\left(\frac{\partial\phi_1}{r_1\partial r_1}\right),
\end{aligned}\right\} \tag{38}$$

where $s = \eta_0 S_0/(\rho^2a^4)$.‡

† Obtained by working in terms of the contravariant components of the stress and rate-of-strain tensors, and later transforming to physical components in the co-ordinate directions.

‡ For Oldroyd's liquid B (equation (4)), s is given by $s = \eta_0^2\lambda_1(\lambda_1-\lambda_2)/(\rho^2a^4)$, and $s = 0$ for the Newtonian liquid.

The associated equation, obtained from (16) and (17), after elimination of p^* and substitution of these stress components, is

$$r_1 \nabla_1^4 \phi_1 = 2r_1^2(1-r_1^2)\cos\psi + 8mr_1^2\cos\psi, \tag{39}$$

where
$$\nabla_1^2 \equiv \left[\frac{\partial^2}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial}{\partial r_1} + \frac{1}{r_1^2}\frac{\partial^2}{\partial\psi^2}\right].$$

The solution of equation (39), satisfying the boundary conditions

$$\partial\phi_1/\partial\psi = \partial\phi_1/\partial r_1 = 0 \quad \text{on} \quad r_1 = 1,$$

is
$$\phi_1 = \{(\tfrac{1}{144} + \tfrac{1}{24}m)\,r_1 - (\tfrac{1}{64} + \tfrac{1}{12}m)\,r_1^3 + (\tfrac{1}{96} + \tfrac{1}{24}m)\,r_1^5 - \tfrac{1}{576}r_1^7\}\cos\psi. \tag{40}$$

Substituting from (40) into the expressions for $p''_{(r_1r_1)1}$, $p''_{(\psi\psi)1}$, etc., we have

$$p''_{(r_1r_1)1} = -[(\tfrac{1}{16} + \tfrac{1}{3}m)\,r_1 - (\tfrac{1}{12} + \tfrac{1}{3}m)\,r_1^3 + \tfrac{1}{48}r_1^5]\sin\psi, \tag{41}$$

$$p''_{(\psi\psi)1} = [(\tfrac{1}{16} + \tfrac{1}{3}m)\,r_1 - (\tfrac{1}{12} + \tfrac{1}{3}m)\,r_1^3 + \tfrac{1}{48}r_1^5]\sin\psi, \tag{42}$$

$$p''_{(\theta\theta)1} = -8mr_1\,\partial w_1/\partial r_1 - s[(\tfrac{1}{6} + m)\,r_1 + (\tfrac{3}{8} + 2m)\,r_1^3 - (\tfrac{3}{4} + 3m)\,r_1^5 + \tfrac{5}{24}r_1^7]\sin\psi, \tag{43}$$

$$p''_{(\psi\theta)1} = \partial w_1/r_1\partial\psi + m[(\tfrac{1}{72} + \tfrac{1}{12}m) + (\tfrac{7}{32} + \tfrac{7}{6}m)\,r_1^2 - (\tfrac{31}{48} + \tfrac{31}{12}m)\,r_1^4 + \tfrac{71}{288}r_1^6]\cos\psi, \tag{44}$$

$$p''_{(r_1\theta)1} = \partial w_1/\partial r_1 + m[(\tfrac{1}{72} + \tfrac{1}{12}m) + (\tfrac{5}{32} + \tfrac{5}{6}m)\,r_1^2 - (\tfrac{11}{48} + \tfrac{11}{12}m)\,r_1^4 + \tfrac{17}{288}r_1^6]\sin\psi, \tag{45}$$

$$p''_{(r_1\psi)1} = [-(\tfrac{1}{16} + \tfrac{1}{3}m)\,r_1 + (\tfrac{1}{6} + \tfrac{2}{3}m)\,r_1^3 - \tfrac{36}{576}r_1^5]\cos\psi. \tag{46}$$

Substitution from (44), (45) into (18), and consideration of only those terms involving L then gives

$$-\frac{\partial\phi_1}{r_1\partial\psi}\frac{\partial}{\partial r_1}(1-r_1^2) = \nabla_1^2 w_1 + m\{(\tfrac{1}{4} + \tfrac{4}{3}m)\,r_1 - (\tfrac{1}{2} + 2m)\,r_1^3 + \tfrac{1}{6}r_1^5\}\sin\psi. \tag{47}$$

Substituting for ϕ_1 from (40) into this equation, we have

$$\nabla_1^2 w_1 = \{-(\tfrac{1}{72} + \tfrac{1}{3}m + \tfrac{4}{3}m^2)\,r_1 + (\tfrac{1}{32} + \tfrac{2}{3}m + 2m^2)\,r_1^3 - (\tfrac{1}{48} + \tfrac{1}{4}m)\,r_1^5 + \tfrac{1}{288}r_1^7\}\sin\psi. \tag{48}$$

The solution of (48) which satisfies the boundary conditions, namely that $w_1 = 0$ when $r_1 = 1$ and w_1 is finite when $r_1 = 0$, is

$$\begin{aligned} w_1 = \tfrac{1}{576}\sin\psi\{(\tfrac{19}{40} + 11m + 48m^2)\,r_1 - (1 + 24m + 96m^2)\,r_1^3 \\ + (\tfrac{3}{4} + 16m + 48m^2)\,r_1^5 - (\tfrac{1}{4} + 3m)\,r_1^7 + \tfrac{1}{40}r_1^9\}. \end{aligned} \tag{49}$$

3. Stream-line projections

The differential equation of any streamline is

$$\frac{dr}{U} = \frac{r\,d\psi}{V} = \frac{(R + r\sin\psi)\,d\theta}{W},$$

which can be written to sufficient accuracy as

$$\frac{dr}{U} = \frac{r\,d\psi}{V} = \frac{Ra^2\,d\theta}{W_0(a^2-r^2)}. \tag{50}$$

Equation (50) is complicated and it is difficult to give a closed expression for the equation of a general streamline; Dean (1927) has pointed out that the precise relation between r, ψ and θ is of little interest, and has drawn attention to the useful projections of stream-lines represented by (r, θ)- and (r, ψ)-relations.

The motion of the liquid is of special simplicity in the central plane of the pipe. At any point on OC, ψ is either $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$; in either case, $\cos\psi$ and V vanish. At any such point the direction of the velocity of the liquid lies in the central plane; hence a particle of liquid once in this plane does not leave it in the subsequent motion. The motion in one half of the pipe is therefore quite distinct from that in the other and the central plane is clearly a plane of symmetry for the motion. The differential equation of the stream lines in the central plane is

$$dr/U = Ra^2 d\theta/W_0(a^2-r^2).$$

Writing $U = -(L\nu/r)\,\partial\phi_1/\partial\psi$ and putting $\sin\psi = 1$, we have

$$\frac{dr}{Rd\theta} = -\frac{\nu a^2 L}{W_0(a^2-r^2)}\left(\frac{\partial\phi_1}{r\partial\psi}\right)_{\sin\psi=1},$$

or $$\frac{dr_1}{Rd\theta} = \frac{L}{na(1-r_1^2)}\{(\tfrac{1}{144}+\tfrac{1}{24}m)-(\tfrac{1}{64}+\tfrac{1}{12}m)r_1^2+(\tfrac{1}{96}+\tfrac{1}{24}m)r_1^4-\tfrac{1}{576}r_1^6\};$$

substitution for L from (20) in this equation gives

$$dr_1/d\theta = \tfrac{1}{288}n(1-r_1^2)(4+24m-r_1^2). \tag{51}$$

It follows from (51) that

$$\theta = \frac{144}{(h^2-1)hn}\log_e\left[\left(\frac{1+r_1}{1-r_1}\right)^h\left(\frac{h-r_1}{h+r_1}\right)\right], \tag{52}$$

where $h^2 = 4+24m$ and where it has been assumed that θ is measured from the point where the streamline crosses the central line $r_1 = 0$. Equation (52) was derived by putting $\sin\psi = 1$ and therefore applies to only those parts of the streamlines in the central plane outside the circle we have called the central line; to obtain the parts inside the circle we write $\sin\psi = -1$, and so reverse the sign of θ in (52).

The other set of equations which are of interest are those giving the movement of liquid elements in relation to the central line. This can be visualized by constructing the projection of a streamline on the section θ = const., taking these projections as sufficiently represented by ϕ_1 = const., where ϕ_1 is given by (40).

Figure 2 shows the paths of particles projected on the cross-section of the pipe, in the cases of an elastico-viscous liquid for which $m = 1$ and of a Newtonian viscous liquid (for which $m = 0$). It can be seen that the form of the projections of the streamlines of the first liquid are not strongly dependent on its elasticity; the positions of the neutral points, where the velocity in the section vanishes, are slightly nearer the outer edge of the pipe in the case of the elastico-viscous liquid, being at $r_1 = 0{\cdot}445$, $\psi = 0$ or π when $m = 1$, compared with $r_1 = 0{\cdot}429$, $\psi = 0$ or π when $m = 0$. The data for the Newtonian liquid are taken from Dean's 1927 paper.

However, elasticity of the type considered could strongly affect the pitch of the spirals in which particles of the liquid move along the pipe; figure 3 illustrates the dependence of the form of the streamlines in the central plane on the parameter m, curves being plotted for $m = 0$, $m = 0{\cdot}2$ and $m = 1{\cdot}0$. The Reynolds number used in the calculations was 63·3, this value being chosen so that the value θ, as given by (52), could be found in degrees by multiplying logarithms to the

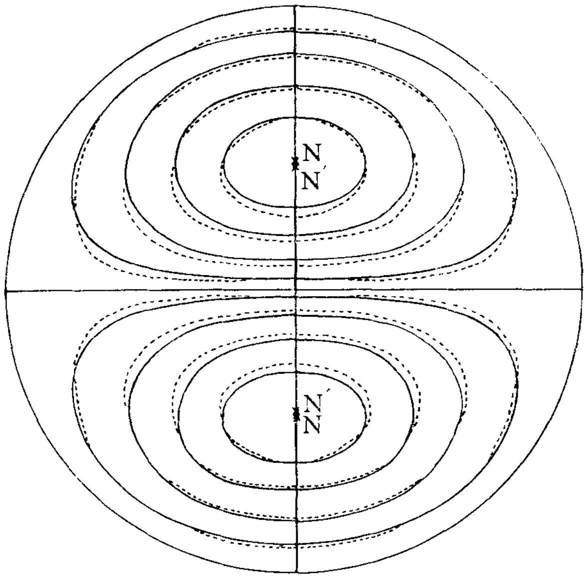

FIGURE 2. Paths of particles projected on the cross-section of the pipe for $m = 1$ (full line) and $m = 0$ (broken line). N and N′ are the neutral points for $m = 1$ and $m = 0$, respectively.

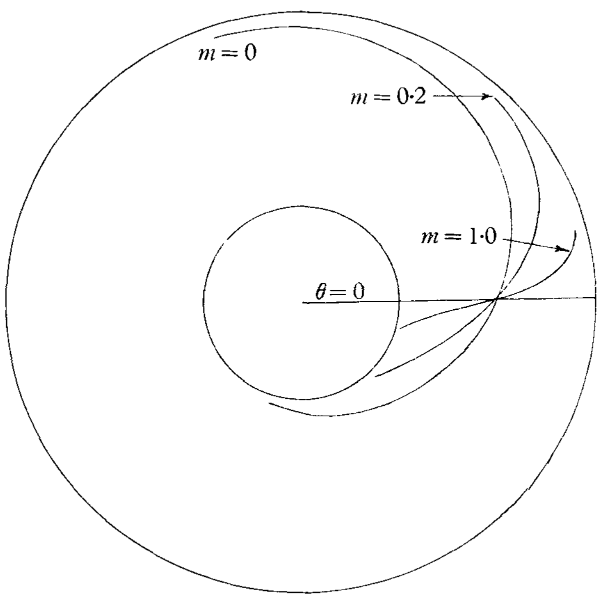

FIGURE 3. The path of a particle in the central plane for various values of m.

base 10 by 50 (cf. Dean 1927); and *for the sake of convenience in drawing* we have assumed that a/R is $\frac{1}{2}$. It is seen from figure 3 that an increase in m leads to a spectacular decrease in the curvature of the streamlines in the central plane.

4. The flux of liquid through the pipe

The rate of flow through the pipe is a constant times

$$\int_0^1 r_1\,dr_1 \int_0^{2\pi} w\,d\psi,$$

and only product and higher-order terms in $\cos\psi$ and $\sin\psi$ contribute to this integral. It follows, since w_1 makes no contribution to this integral (see equation (49)), that the flux through the pipe is independent of the curvature, to the first-order approximation; in order to study the variation of flux with curvature, therefore, we must introduce terms of higher order. This is carried out in the following paragraphs.

Although the relevant equations are complicated, it is possible to simplify the working when the variation of flux with L^2 only is required; the method of simplification used is an extension of the method used already by Dean (1928).

For mathematical convenience, it is necessary to restrict the discussion to liquids with short memories, i.e. liquids with short relaxation times. We shall neglect terms involving $(t-t')^q$, $q \geqslant 2$ in equations (30) to (32). This approximation is equivalent to neglecting terms involving

$$\int_0^\infty \tau^q N(\tau)\,d\tau \quad (q \geqslant 2)$$

in comparison with those involving

$$\int_0^\infty \tau N(\tau)\,d\tau \quad \text{and} \quad \int_0^\infty N(\tau)\,d\tau.$$

Such an approximation would be justified, for example, in the case of the dilute polymer solutions investigated by Oldroyd, Strawbridge & Toms (1951).

Going through the same procedure as before, except that now terms involving $(t-t')^q$ $(q \geqslant 2)$ are ignored, it can be shown that

$$p''_{(r_1r_1)2} = 2\frac{\partial u_2}{\partial r} - 2m\left[u_1\frac{\partial^2 u_1}{\partial r_1^2} + \frac{v_1}{r_1}\frac{\partial^2 u_1}{\partial r_1\,\partial\psi} - 2\left(\frac{\partial u_1}{\partial r_1}\right)^2 - \frac{1}{r_1^2}\left(\frac{\partial u_1}{\partial\psi}\right)^2 - \frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\frac{\partial u_1}{\partial\psi}\right], \tag{53}$$

$$\begin{aligned} p''_{(\psi\psi)2} = {} & 2\left[\frac{1}{r_1}\frac{\partial v_2}{\partial\psi} + \frac{u_2}{r_1}\right] - 2m\left[r_1^2 u_1\frac{\partial}{\partial r_1}\left[\frac{1}{r_1^3}\frac{\partial v_1}{\partial\psi} + \frac{u_1}{r_1^3}\right] + \frac{v_1}{r_1^2}\frac{\partial}{\partial\psi}\left[\frac{\partial v_1}{\partial\psi} + u_1\right]\right. \\ & \left. - r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\left[\frac{\partial u_1}{r_1\,\partial\psi} + r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\right] - \frac{2}{r_1^2}\frac{\partial v_1}{\partial\psi}\left(\frac{\partial v_1}{\partial\psi} + u_1\right)\right], \end{aligned} \tag{54}$$

$$p''_{(\theta\theta)2} = 2m\left[\left(\frac{\partial w_1}{\partial r_1}\right)^2 - 4r_1\frac{\partial w_1}{\partial r_1} + \frac{1}{r_1^2}\left[\frac{\partial w_1}{\partial\psi}\right]^2\right], \tag{55}$$

$$\begin{aligned} p''_{(r_1\theta)2} = {} & \frac{\partial w_2}{\partial r_1} + m\left[2u_2 - u_1\frac{\partial^2 w_1}{\partial r_1^2} - \frac{v_1}{r_1}\frac{\partial^2 w_1}{\partial r_1\,\partial\psi} - 6r_1\frac{\partial u_2}{\partial r_1} + 3\frac{\partial w_1}{\partial r_1}\frac{\partial u_1}{\partial r_1}\right. \\ & \left. + \frac{2}{r_1^2}\frac{\partial w_1}{\partial\psi}\frac{\partial u_1}{\partial\psi} + \frac{\partial w_1}{\partial\psi}\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\right], \end{aligned} \tag{56}$$

$$p''_{(r_1\psi)2} = \left[\frac{\partial u_2}{r_1\partial\psi} + r_1\frac{\partial}{\partial r_1}\left(\frac{v_2}{r_1}\right)\right] - m\left\{r_1 u_1 \frac{\partial}{\partial r_1}\left[\frac{\partial u_1}{r_1^2\partial\psi} + \frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\right]\right.$$
$$+\frac{v_1}{r_1}\frac{\partial}{\partial\psi}\left[\frac{\partial u_1}{r_1\partial\psi} + r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\right] - 2r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\frac{\partial u_1}{\partial r_1}$$
$$\left. -\left(\frac{\partial u_1}{\partial r_1} + \frac{1}{r_1}\frac{\partial v_1}{\partial\psi}\right)\left(\frac{\partial u_1}{r_1\partial\psi} + r_1\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right)\right) - 2\frac{\partial u_1}{r_1\partial\psi}\left[\frac{\partial v_1}{r_1\partial\psi} + \frac{u_1}{r_1}\right]\right\}, \quad (57)$$

$$p''_{(\psi\theta)2} = \frac{1}{r_1}\frac{\partial w_2}{\partial\psi} - m\left\{r_1 u_1 \frac{\partial}{\partial r_1}\left(\frac{\partial w_1}{r_1^2\partial\psi}\right) + \frac{v_1}{r_1^2}\frac{\partial^2 w_1}{\partial\psi^2} + 2\frac{\partial u_2}{\partial\psi} + 4r_1^2\frac{\partial}{\partial r_1}\left(\frac{v_2}{r_1}\right) - \frac{1}{r_1}\frac{\partial w_1}{\partial r_1}\frac{\partial u_1}{\partial\psi}\right.$$
$$\left. - r_1\frac{\partial w_1}{\partial r_1}\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right) - r_1\frac{\partial w_1}{\partial r_1}\frac{\partial}{\partial r_1}\left(\frac{v_1}{r_1}\right) - \frac{2}{r_1}\frac{\partial w_1}{\partial\psi}\left[\frac{\partial v_1}{r_1\partial\psi} + \frac{u_1}{r_1}\right] - \frac{1}{r_1^2}\frac{\partial v_1}{\partial\psi}\frac{\partial w_1}{\partial\psi}\right\}. \quad (58)$$

When account is taken of the functional forms of ϕ_1 and w_1, and use is made of (37), we have from (53), (54) and (57)

$$p''_{(r_1r_1)2} = 2\left(\frac{\partial\phi_2}{r_1^2\partial\psi} - \frac{\partial^2\phi_2}{r_1\partial r_1\partial\psi}\right) + F_1\cos 2\psi + F_2,$$

$$p''_{(\psi\psi)2} = 2\left(\frac{\partial^2\phi_2}{r_1\partial r_1\partial\psi} - \frac{\partial\phi_2}{r_1^2\partial\psi}\right) + G_1\cos 2\psi + G_2,$$

$$p''_{(r_1\psi)2} = \left[-\frac{\partial^2\phi_2}{r_1^2\partial\psi^2} + r_1\frac{\partial}{\partial r_1}\left(\frac{\partial\phi_2}{r_1\partial r_1}\right)\right] + H_1\sin 2\psi,$$

where F_1, F_2, G_1, G_2 and H_1 are functions of r_1 which need not be more particularly specified; substituting these stress components into the associated equation of motion obtained from (16) and (17) after elimination of p^*, we find that the equation for ϕ_2 is of the form

$$\nabla_1^4\phi_2 = I_1\sin 2\psi,$$

whence
$$\phi_2 = J_1\sin 2\psi, \quad (59)$$

where I_1 and J_1 are functions of r_1 only. Similarly, from (56) and (58),

$$p''_{(r_1\theta)2} = \partial w_2/\partial r_1 + T_1(r_1)\cos 2\psi + T_2(r_1), \quad (60)$$

$$p''_{(\psi\theta)2} = \partial w_2/r_1\partial\psi + K_1(r_1)\sin 2\psi. \quad (61)$$

Substitution of these stresses in the associated equation of motion obtained from (18) gives an equation of the form

$$r_1\nabla_1^2 w_2 = -\left(\frac{T_1}{r_1} + T_1' + \frac{2K_1}{r_1}\right)\cos 2\psi - \left(\frac{T_2}{r_1} + T_2'\right)$$
$$+\frac{\partial\phi_1}{\partial r_1}\frac{\partial w_1}{\partial\psi} - \frac{\partial\phi_1}{\partial\psi}\frac{\partial w_1}{\partial r_1} - \frac{\partial\phi_2}{\partial\psi}\frac{\partial}{\partial r_1}(1-r_1^2), \quad (62)$$

where the dash denotes differentiation with respect to r_1; whence w_2 is of the form $N_1(r_1)\cos 2\psi + N_2(r_1)$. The first of these terms need not be evaluated since it does not affect the flux. Since ϕ_2 and the second term in the expression for $p''_{(\psi\theta)2}$ give a contribution to this term only, it follows that J_1 and K_1 need not be evaluated. Thus the problem is reduced to finding T_2 in the expression for $p''_{(r_1\theta)2}$, and then that part of w_2 which is a function of r_1 only ($\overline{w}_2$, say).

From (37), (40), (49), (56), (60), we have

$$T_2 = \{\tfrac{1}{2}m/(576)^2\}\{-2\cdot 2r_1 + 68\cdot 4r_1^3 - 166\cdot 8r_1^5 + 164r_1^7 - 81r_1^9 + 19\cdot 2r_1^{11} - 1\cdot 6r_1^{13}\}, \quad (63)$$

where we have neglected terms of order m^2 to be consistent with our present confinement of the discussion to liquids with short memories. Consideration of terms involving r_1 only in (62) gives

$$d(r_1 d\bar{w}_2/dr_1)/dr_1 = \tfrac{1}{2}(M_3' N_3 + M_3 N_3') - d(r_1 T_2)/dr_1, \quad (64)$$

where it has been assumed that $\phi_1 = M_3 \cos\psi$, $w_1 = N_3 \sin\psi$. Equation (64) integrates to give

$$r_1 d\bar{w}_2/dr_1 = \tfrac{1}{2}M_3 N_3 - r_1 T_2, \quad (65)$$

the constant of integration being zero since w_2 is finite at $r_1 = 0$. Substituting for the quantities M_3, N_3 and T_2 from (40), (49), (63) into (65) and integrating, we have

$$\begin{aligned}\bar{w}_2 = \tfrac{1}{2}(576)^{-2}\{&-[0\cdot 1839 + 3\cdot 1897m] + [0\cdot 95 + 28\cdot 8m]r_1^2 - [2\cdot 0687 + 77\cdot 55m]r_1^4 \\ &+ [2\cdot 475 + 98\cdot 366m]r_1^6 - [1\cdot 778 + 67\cdot 625m]r_1^8 + [0\cdot 785 + 25\cdot 86m]r_1^{10} \\ &- [0\cdot 2062 + 5\cdot 0333m]r_1^{12} + [0\cdot 0286 + 0\cdot 3714m]r_1^{14} - 0\cdot 0016r_1^{16}\}, \quad (66)\end{aligned}$$

where the arbitrary constant of integration has been chosen so that $\bar{w}_2$ vanishes when $r_1 = 1$.

The flux through the pipe per second is given by

$$\begin{aligned}F_c &= \int_0^a r\,dr \int_0^{2\pi} W\,d\psi \\ &= 2\pi W_0 a^2 \int_0 r_1[(1 - r_1^2) + L^2\bar{w}_2 + O(L^4)]\,dr_1.\end{aligned}$$

Substituting from (66) and evaluating the integral, we have

$$F_c = F_s[1 - (\tfrac{1}{576}L)^2(0\cdot 03058 - 0\cdot 06426m)], \quad (67)$$

where F_s ($= W_0 \pi a^2/2$) is the value of the flux per second through a straight pipe of the same cross-section under the same dynamic conditions. It is observed that the presence of elasticity in the liquid decreases the dependence of the flux through the pipe on the curvature of the pipe, to second order in the curvature.†

Concluding, therefore, we see that the main effect of elasticity of the type considered on the flow of an elastico-viscous liquid through a curved pipe under a pressure gradient is to decrease the curvature of the stream lines in the central plane and also to increase the volume of fluid flowing through the pipe in unit time.

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† The reader is reminded that equation (67) is valid only for small values of m.

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